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One-Loop Superstring Cosmology and the Non-Singular Universe

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Abstract

We study the cosmological implications of the one-loop terms in the string expansion. In particular, we find non-singular solutions which interpolate between a contracting universe and an expanding universe, and show that these solutions provide a mechanism for removing the initial conditions problem peculiar to spatially closed FRW cosmologies. In addition, we perform numerical calculations to show that the non-singular cosmologies do not require a careful choice of initial conditions, and estimate the likely magnitude of higher order terms in the string expansion.

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1 Introduction

The study of cosmology leads inexorably to an epoch in which the energy density of the universe approaches the Planck scale. Superstring theory is the leading candidate for a description of physics at the Planck scale, but has not developed to the point where the full theory may be employed to construct a detailed cosmology. Despite this, the generic properties of superstring models can be extracted and their cosmological consequences investigated. In doing so, the cosmologist simultaneously hopes to find mechanisms for resolving the various problems that remain after the introduction of the inflationary scenario, and to test string theory in the high energy laboratory provided by the early universe.

Two distinct approaches have been taken to the study of string cosmology. Firstly, many cosmologists approximate the full string theory with the first terms of the perturbative string expansion [1, 2, 3]. The non-perturbative features of string theory are implicitly discarded, so this approach breaks down at the highest energies. However, since Einstein gravity and conventional particle physics agree well with experiment, a successful superstring theory must reduce to these theories in the low energy limit. Consequently, it is reasonable to expect that at energy scales when stringy effects first become readily apparent it will be possible to approximate the full string theory by an effective theory comprising the usual Einstein-Hilbert action together with higher order corrections. In particular, much of the effort to date has focussed upon the lowest order, or tree-level action [4]-[21].

Gasperini, Veneziano and others have adopted an alternative strategy, which they have dubbed the "pre-big-bang scenario" [22, 23]. This is a non-singular cosmology, and it attempts to alleviate many of the problems associated with the "standard" big bang by appealing to the symmetries of the full superstring action, rather than a perturbative limit. The fundamental requirement for such a model is the existence of non-singular or "branch-changing" [11, 17] solutions which smoothly interpolate between a contracting universe and an expanding one without passing through a singularity.

The term branch-change originally referred to a non-singular transition between two solutions of a simple tree-level string cosmology. The solutions are

distinguished by a sign choice and are related to one another by a duality transformation. Moreover, this discussion was largely based in the (tree-level) string frame, which is related to the usual Einstein frame of general relativity by a conformal transformation. A universe which is expanding in the string frame may be contracting in the Einstein frame, and it is therefore dangerous to simply identify the two branches with "expansion" and "contraction". However, in general the $(-)$ branch evolves away from a past singularity, while the $(+)$ branch represents evolution towards a future singularity [24]. The type of branch-change required by the pre-big-bang scenario is a non-singular transition from the $(+)$ to the $(-)$ branch. The converse, a solution which moves from $(-)$ to $(+)$ is entirely unremarkable, as it corresponds to the generic evolution of a spatially closed FRW universe.

Since string theory is expected to remove, or at least tame, the infinities inherent in other models of fundamental physics, it is reasonable to expect that it will remove the singularity represented by the big bang in conventional cosmology. At tree-level in the superstring action, though, the possibility of classical branch-changing solutions to the classical equations of motion has been virtually excluded [21, 24, 25], although quantum cosmology may provide a mechanism for allowing a transition between the two tree-level branches [26, 27]. However, the tree-level action is only the lowest order term in the full string loop expansion, so consequently in this paper we explore the cosmological solutions that arise when one loop contributions from the dilaton and modulus terms are included. We show that in this case, the low-energy limit of string theory naturally leads to a "bouncing universe", in contrast to the tree-level limit.

Previously, Antoniadis, Rizos and Tamvakis [28] have studied the equations of motion derived from the same action with a spatially flat background and found some non-singular solutions. However, in this case the scale factor increases monotonically from a (non-zero) constant value, which differs from the branch-changing solutions envisaged by the pre-big-bang scenario, as there is no transition from contraction to expansion. We extend this system to the case where the spatial hypersurfaces are allowed to have non-zero curvature. In doing so we find "bouncing" solutions and investigate the range of initial conditions over which they occur. Since we do not directly employ the

duality symmetries of the superstring action in our analysis, we do not explicitly identify these bouncing solutions with the branch-changing solutions of the pre-big-bang scenario. However, from a phenomenological perspective, a bouncing solution resembles a successful branch-change, as in both cases the universe is apparently evolving towards a singularity in the distant past and away from it in the distant future.

We also consider the particular initial conditions problem faced by a spatially closed FRW universe, which in the absence of fine-tuning will typically have a lifetime on the order of the Planck scale [29]. This problem persists in the presence of inflation: although inflation can begin at the Planck scale there is no guarantee that it will do so. Here we show that the contribution from the one-loop terms can allow a closed universe to grow arbitrarily large, ensuring that the universe will survive long enough for inflation to begin.

Finally, we numerically integrate the equations of motion for a wide range of initial conditions, to demonstrate that the bouncing solutions do not require a highly restrictive choice of parameters. We also argue on the basis of dimensional analysis that many of the bouncing solutions do not evolve into a region where the perturbative expansion is likely to break down, which makes it plausible that these solutions will persist if higher loop terms are added to the action.

2 Action and Equations of Motion

We take as our starting point the one-loop superstring action [28, 30, 31],

$$S = \int d^4x \sqrt{-g} \left\{ \frac{1}{2}R - \frac{1}{4}(D\phi)^2 - \frac{3}{4}(D\sigma)^2 + \frac{1}{16}\left[\lambda e^{\phi} - \delta\xi(\sigma)\right] R^2_{\rm GB} \right\}, \tag{1}$$

which contains contributions from the Ricci scalar R, the dilaton ϕ and a modulus field σ. The Gauss-Bonnet combination, $R^2_{\rm GB}$, is

$$R^2_{\rm GB} = R_{\mu\nu\kappa\lambda}R^{\mu\nu\kappa\lambda} - 4R_{\mu\nu}R^{\mu\nu} + R^2. \tag{2}$$

Terms which appear at one loop in the string expansion but vanish when the metric has the Robertson-Walker form have been dropped from the above action. We have adopted the same conventions and notation as Antoniadis, Rizos and Tamvakis with the exception of the metric signature, which we have

set to $(-,+,+,+)$, and our units correspond to the gravitational constant, $G = 1/8\pi$. The coefficient λ is positive and determined by the four dimensional string coupling. The sign of δ is determined by the relative numbers of chiral, vector and spin-3/2 massless supermultiplets, and is proportional to the four dimensional trace anomaly of the $N = 2$ sector of the theory. It will be important that δ can take both positive and negative values. The potential, $\xi(\sigma)$ is defined in terms of the Dedekind η function,

$$\xi(\sigma) = \ln\left[2e^{\sigma}\eta^4(ie^{\sigma})\right] \tag{3}$$

where η is [32]

$$\eta(\tau) = q^{1/12}\prod_{n=1}^{\infty}(1-q^{2n}), \qquad q = e^{i\pi\tau}. \tag{4}$$

Anticipating that first and second derivatives of ξ will appear in the equations of motion, we note that

$$\xi_\sigma(\sigma) = 1 - \frac{\pi e^{\sigma}}{3} + 8\pi e^{\sigma}\sum_{n=1}^{\infty}\frac{ne^{-2n\pi e^{\sigma}}}{1-e^{-2n\pi e^{\sigma}}} \tag{5}$$

where the subscript denotes differentiation with respect to σ. Despite its appearance, this is an odd function of σ. Furthermore, for large $|\sigma|$

$$\xi_\sigma \approx -\frac{2\pi}{3}\sinh(\sigma) \tag{6}$$

closely approximates the exact expression, equation (5), and is also antisymmetric under $\sigma \to -\sigma$. The accuracy of this approximation for $\sigma \approx 0$ could be improved by adding terms of the form $c_n\sigma^n$ for $n = 1, 3, 5\ldots$ (since ξ_σ is odd, even powers of σ will not appear) with the c_n chosen so that the approximation reproduces the first few terms in the Taylor expansion of ξ_σ about $\sigma = 0$. However, in practice adding these corrections does not alter the qualitative properties of the solutions obtained numerically, so for simplicity we have worked exclusively with equation (6).

Previously Antoniadis, Rizos and Tamvakis examined the cosmological solutions for this system that have a spatially flat Robertson Walker metric. We extend their work to include the possibility that the spatial hypersurfaces have non-zero curvature, and so the appropriate ansatz for the line element is

$$ds^2 = -dt^2 + e^{2\omega(t)}\left[\frac{1}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right]. \tag{7}$$

We therefore derive the equations of motion

$$3\left(\dot{\omega}^2 + ke^{-2\omega}\right)\left(1 + 8\dot{f}\dot{\omega}\right) - \tfrac{1}{4}\dot{\phi}^2 - \tfrac{3}{4}\dot{\sigma}^2 = 0 \tag{8}$$

$$2\left(\ddot{\omega} + \dot{\omega}^2\right)\left(1 + 8\dot{f}\dot{\omega}\right) + \left(\dot{\omega}^2 + ke^{-2\omega}\right)\left(1 + 8\ddot{f}\dot{\omega}\right) + \tfrac{1}{4}\dot{\phi}^2 + \tfrac{3}{4}\dot{\sigma}^2 = 0, \tag{9}$$

$$\ddot{\phi} + 3\dot{\omega}\dot{\phi} - 2\tfrac{df}{d\phi}R^2_{\rm GB} = 0, \tag{10}$$

$$\ddot{\sigma} + 3\dot{\omega}\dot{\sigma} - \tfrac{2}{3}\tfrac{df}{d\sigma}R^2_{\rm GB} = 0, \tag{11}$$

where a dot denotes differentiation with respect to t. The Gauss-Bonnet term is

$$R^2_{\rm GB} = 24\left(\ddot{\omega} + \dot{\omega}^2\right)\left(\dot{\omega}^2 + \frac{k}{e^{2\omega}}\right) \tag{12}$$

and f is defined to be

$$f = \frac{1}{16}\left[\lambda e^{\phi} - \delta\xi(\sigma)\right]. \tag{13}$$

If f vanishes, the system reduces to two free, minimally coupled, scalar fields in a Robertson-Walker universe.

The equations above consist of three second order equations and a constraint, giving a total of five degrees of freedom. In order to facilitate the analysis we isolate each of the second derivative terms (remembering that $\ddot{f}$ implicitly contains $\ddot{\phi}$ and $\ddot{\sigma}$), on the left hand side, giving the following system

$$\begin{aligned}
\ddot{\omega} &= -\dot{\omega}^2 - \left(\dot{\omega}^2 + \frac{k}{e^{2\omega}}\right)\chi, && (14)\\
\ddot{\phi} &= -3\dot{\omega}\dot{\phi} - 3\lambda e^{\phi}\left(\dot{\omega}^2 + \frac{k}{e^{2\omega}}\right)^2\chi, && (15)\\
\ddot{\sigma} &= -3\dot{\omega}\dot{\sigma} + \delta\xi_\sigma\left(\dot{\omega}^2 + \frac{k}{e^{2\omega}}\right)^2\chi, && (16)
\end{aligned}$$

where

$$\chi = \frac{8 + \lambda\dot{\phi}^2 e^{\phi} - \delta\dot{\sigma}^2\xi_{\sigma\sigma}}{4 + 2(\lambda\dot{\phi}e^{\phi} - \delta\dot{\sigma}\xi_\sigma)\dot{\omega} + (\dot{\omega}^2 + ke^{-2\omega})^2(3\lambda^2 e^{2\phi} + \delta^2\xi_\sigma^2)}. \tag{17}$$

If we wish, we can eliminate any one of the variables by inserting the constraint. However, it will be more convenient to work with the equations as they are given above, both in the following section when we consider the asymptotic form of the solutions and in Section 4, where the constraint will allow us to the check the accuracy of our numerical solutions.

3 Asymptotic Solutions

In general this system of equations must be solved numerically. However, considerable insight into the cosmological properties of this model can be gained from analytic considerations alone. In particular, the existence of a bouncing solution requires that the scale factor to pass through a (non-zero) minimum value and be singularity free. Hence we begin our investigation by considering the possible extrema of ω, ϕ and σ.

When the scale factor, $a = e^{\omega}$, passes through a local minimum, its second derivative, $\ddot{a}$, must be positive. When $k = 0$ the constraint, equation (8), with $\dot{\omega} = 0$ requires $\dot{\sigma} = \dot{\phi} = 0$ as well. This is an exact static solution to the equations of motion where the values of ω, σ and ϕ are all arbitrary constants. The non-singular solutions of Antoniadis, Rizos and Tamvakis can be regarded as the consequence of making a small deviation from this solution in the distant past. If the spatial hypersurfaces have negative curvature, $k = -1$ and the constraint cannot be satisfied when $\dot{\omega} = 0$. Consequently, the scale factor is again monotonic, but the constant solution no longer exists.

When the spatial hypersurfaces have positive curvature, so that $k = 1$, the constraint can be satisfied with $\dot{\omega} = 0$ and $\dot{\phi}$, $\dot{\sigma} \neq 0$. Thus, in this case, the scale factor may pass through an extremal value. Since the first two terms in the denominator of χ are equal to $4(1 + 8\dot{f}\dot{\omega})$ it follows from the constraint that if $k = 1$ the denominator of χ is positive, so

$$\mathrm{Sign}[\ddot{\omega}|_{\dot{\omega}=0}] = \mathrm{Sign}[-(8 + \lambda\dot{\phi}^2 e^{\phi} - \delta\dot{\sigma}^2 \xi_{\sigma\sigma})] \tag{18}$$

While λ is physically restricted to positive values, δ can take any real value. Since $\xi_{\sigma\sigma}(\sigma) < 0$ for all σ, if $\delta \geq 0$, $\ddot{\omega}$ will be negative at an extremum of ω, but can take either sign if $\delta < 0$. This immediately proves the existence of solutions for which $k = 1$ and $\delta < 0$ where the scale factor possesses a local minimum, which is a prerequisite for a successful bounce. However, this does not establish the existence of globally non-singular solutions, since it does not guarantee that such a solution will always be non-singular. For instance, when the tree level action contains contributions from the spatial curvature and an axion the Einstein frame scale factor may pass through several local minima, but all solutions contain at least one singularity [21].

Now consider the asymptotic form of solutions to the equations of motion. We focus on $k = 1$, which is the only choice that can lead to a bouncing solution and investigate the properties of a universe which is expanding at large positive times

$$\begin{array}{rll} \text{Type I} & \ddot{a}(t) > \epsilon, & t > T, \\ \text{Type II} & \ddot{a}(t) < -\epsilon, & t > T, \\ \text{Type III} & \ddot{a}(t) \to 0, & t \to \infty, \end{array} \tag{19}$$

where ϵ and T are both positive numbers. A solution which is contracting at large, negative times corresponds to the time-reverse of this case and does not need to be examined separately.

Since the defining condition for inflation is that $\ddot{a} > 0$ [33], a solution of Type I inflates forever and so the curvature terms will be negligible at late times. However, this would imply the existence of solutions for $k = 0$ that are inflationary at late times, in contradiction with the results of Antoniadis, Rizos and Tamvakis, so it follows that there are no expanding, asymptotic solutions with the form of Type I. Conversely, in the case of Type II, the curvature term will dominate the $\dot{\omega}^2$ term where they appear together in the equations of motion, and the scale factor will eventually pass through a maximum. Thus there are no solutions which expand indefinitely where the scale factor has the generic form of Type II. Note that if the scale factor does evolve through a local maximum, it may either make a non-singular transition to a subsequent stage of expansion or collapse to a singularity.

Finally, consider possible asymptotic solutions of Type III. Since the Gauss-Bonnet term vanishes if $\ddot{a}$ is identically zero, choosing $a(t) = a_0 t$ is not a viable candidate for a late time solution that involves a non-trivial contribution from the one-loop terms. Consequently, we need to consider solutions with an asymptotic form like $a = e^{\omega} \to a_1 t + a_2 t(\ln t)^m$ which are of Type III but allow for the possibility of a non-zero Gauss-Bonnet combination.

Dropping the contributions from the dilaton terms, make the following substitution, where $\tau = \ln t$

$$A(t) = \frac{a(t)}{t} = a_1 + a_2 \tau^m \tag{20}$$

$$s(t) = \frac{e^{\sigma}}{t^2} = s_1 + s_2 \tau^n \tag{21}$$

which allows us to write equations (8) and (11) as

$$\left[\left(1+\frac{A'}{A}\right)^2+\frac{1}{A^2}\right]\left[4+2\Delta(2s+s')\left(1+\frac{A'}{A}\right)\right]-\left(2+\frac{s'}{s}\right)=0 \qquad (22)$$

$$\frac{s''}{s}=-\left(2+3\frac{A'}{A}\right)\left(2+\frac{s'}{s}\right)+\Delta s\left(\frac{A''}{A}+\frac{A'}{A}\right)\left[\left(1+\frac{A'}{A}\right)^2+\frac{1}{A^2}\right] \qquad (23)$$

where $\Delta = \pi\delta/3$, a dash denotes differentiation with respect to τ and we have assumed that $\sigma \gg 1$ so that $\xi_\sigma \propto e^\sigma$. This involves no loss of generality, since the full equations of motion are invariant under the transformation $\sigma \to -\sigma$.

With the ansatz, equations (20) and (21), we can systematically expand in powers of τ. There are a number of subcases, corresponding to $a_{1,2}$, $s_{1,2}$, m and n being either positive or negative, subject to the overall requirement that $A(t)$ and $S(t)$ are positive. For $k = 1$ and $\delta < 0$ (restricting attention to the possible bouncing solutions) we can show that the only asymptotic solution of this type is:

$$A = \frac{1}{\sqrt{8\tau}}, \qquad (24)$$

$$s = \frac{-1}{\Delta}\left(1+\frac{5}{8\tau}\right). \qquad (25)$$

If $\phi \ll |\sigma|$ and the scale factor is given approximately by equation (24) then the one-loop terms have a negligible effect in equation (10) and it follows that in the asymptotic regime the dilaton ϕ is effectively constant. As written above, this asymptotic form describes an expanding universe as $t \to \infty$. A universe which is contracting as $t \to -\infty$ corresponds to the time reversal of this solution.

We now have the ingredients we need to construct a universe that can be arbitrarily large in the distant past, contract to a non-zero minimum size and then make a smooth transition to expansion, after which it may grow indefinitely. A specific numerical solution of the equations of motion which exhibits these properties is shown in Figure 1. The key ingredient in these solutions is the presence of the modulus field σ and its associated one-loop terms. If either $\delta \geq 0$, or the one-loop terms are simply absent from the action, then the only type of extremum that the scale factor, $a(t)$, can possess is a local maximum, and a bounce does not occur.

An analogous result can be found for $\delta > 0$, but in this case no bounce is possible. Furthermore, any asymptotic solution to the scale factor-modulus system for positive δ will be equivalent to a similar solution to the scale factor-dilaton system in which $\phi \to \infty$, although this implies that the system evolves into the strong-coupling region and such a solution is therefore unphysical.

Solutions of this type extend the previous work of Antoniadis, Rizos and Tamvakis, further demonstrating that the singularity problem is less acute in cosmological models based on the one-loop string action than at tree-level or in classical, Einstein gravity. This provides empirical support for the hope that a fully non-perturbative string cosmology will resolve the singularity problem entirely.

Physically, the key ingredient in the non-singular solutions is the one-loop term that couples the Gauss-Bonnet combination and the modulus; without this there are no non-singular solutions. However, the sign of this coupling is determined by δ and it is only for $\delta < 0$ that the one-loop modulus terms can cause a bounce. Thus, while have established that adding one-loop terms to the low energy string action introduces non-trivial non-singular solutions and significantly softens the conclusions reached at tree-level, these corrections do not imply a complete absence of singularities.

The new class of non-singular cosmologies found here shows that string theory has the ability to resolve the curvature problem typically associated with a closed, FRW universe, which has a typical lifetime (and maximum size) on the order of the Planck scale. Inflation can easily solve the analogous problem in an open universe, which will expand indefinitely, whereas in a spatially closed universe inflation is typically only successful if it begins at the Planck scale. However, we have shown that the presence of the one-loop modulus terms permit a closed FRW universe to expand indefinitely. This is not sufficient to solve the flatness problem, but these solutions provide a mechanism for ensuring that the universe lasts long enough and the energy density becomes low enough for inflation to begin without the need for any additional constraints in a $k = 1$ FRW universe. This is a property of the asymptotic form of the solutions, and solutions which contain an initial singularity may still expand indefinitely. Furthermore, the analogous asymptotic form for $\delta > 0$ also allows indefinite expansion, even though a non-singular, bouncing solution is

impossible in this case.

Solutions of the type displayed in Figure 1 strongly resemble the type of branch-change envisaged by the pre-big-bang scenario. Unlike the non-singular solutions that exist when $k = 0$, the scale factor is not monotonic. As $t \to \pm\infty$ the dilaton is fixed, and the Einstein and string frame scale factors differ only by a multiplicative constant. Hence there is no ambiguity in associating a solution which is contracting in the string frame at early times to one that is contracting in the Einstein frame, as there might be if the dilaton were evolving rapidly.

We can also gain some insight into the behavior of the dilaton and modulus fields. Consider equations (15) and (16), and recall that the denominator of χ is positive for $k = 1$, so

$$\begin{aligned} \mathrm{Sign}[\ddot{\phi}\big|_{\dot{\phi}=0}] &= \mathrm{Sign}\left[-(8 - \delta\dot{\sigma}^2\xi_{\sigma\sigma})\right], && (26) \\ \mathrm{Sign}[\ddot{\sigma}|_{\dot{\sigma}=0}] &= \mathrm{Sign}\left[\delta\xi_\sigma(\sigma)\right], && (27) \end{aligned}$$

since $\lambda > 0$ on physical grounds. Thus when $\delta < 0$, which is the case we are most interested in, the dilaton can possess both maxima and minima and, at least in principle, may undergo several oscillations before the solution becomes established in one of the asymptotic regimes. However, any extremal value of σ when $\sigma < 0$ is a local maximum, and any extremal value when $\sigma > 0$ is a local minimum. Thus when $\delta < 0$, σ can have at most one extremum as it evolves from one asymptotic regime to the other.

When the scale factor a passes through its minimum value, $\ddot{a} > 0$. As this is the minimal requirement for the existence of inflation, it follows that in a pedantic sense a bouncing solution is also inflationary. Obviously, the astrophysical constraints that a successful inflationary model must satisfy are much more demanding than simply requiring that the scale factor undergo positive acceleration. At late times the asymptotic form of the solution approaches the borderline condition between inflationary and regular growth, as $\ddot{a} \to 0$, which resembles the coasting solutions that have been discussed in the context of standard inflationary cosmology [34].

In the next section we use numerical techniques to investigate the generality of these non-singular solutions. However, note that while the scale factor is

monotonic if $k=-1$, there is a simple exact solution

$$\begin{aligned} \omega &= \omega_0 + \ln t, && (28) \\ \phi &= \phi_0, && (29) \\ \sigma &= \sigma_0, && (30) \end{aligned}$$

which describes an empty, curvature dominated universe, and $\ddot{a} \equiv 0$. However, unlike the linearly expanding solution found for $k=1$, this solution holds when λ and δ are zero and the Gauss-Bonnet combination will vanish exactly, so it is not related to the one-loop terms in the string action.

4 Numerical Results

In the previous section we displayed a particular example of a non-singular universe, associated with the one-loop terms in the perturbative expansion of the superstring action. In this section we estimate of the likely impact of higher loop terms and investigate the range of initial conditions that give rise to a non-singular universe.

While it is simple to determine which solutions are singular and which are non-singular, the more qualitative distinction between a solution which comes "close" to the Planck scale, and one where the bounce occurs at a considerably lower energy scale is obviously important. One way to quantify this is through the maximum values of the scalar quantities that can be constructed from the metric curvature, such as R, $R_{\mu\nu}R^{\mu\nu}$, $R_{\mu\nu\kappa\lambda}R^{\mu\nu\kappa\lambda}$ or various higher order combinations. In particular, Brandenberger, Mukhanov and others [35, 36] have constructed models that ensure these quantities remain sub-Planckian for all homogenous and isotropic cosmological solutions. The action considered by us does not have this property since it does admit some singular solutions, for which the curvature invariants will exceed any given finite value. However, the curvature invariants derived from the non-singular solutions are typically sub-Planckian. As a specific example, the value of $R_{\mu\nu\kappa\lambda}R^{\mu\nu\kappa\lambda}$ for the solution of Figure 1 is plotted in Figure 2.

While it is reassuring that the curvature terms do not exceed the Planck scale, the dilaton and modulus fields couple directly to higher-order curvature terms, and if the impact of higher terms in the perturbative expansion is to

be small then these combinations must be sub-Planckian as well. Without a detailed calculation at two-loop order and beyond, it is impossible to state definitively whether the non-singular solutions found at one-loop will persist if higher order terms are added to the action. However, on dimensional grounds the two-loop terms can be expected to be smaller than the one-loop Gauss-Bonnet term provided that $\dot{\omega}$, $\dot{\phi}$ and $\dot{\sigma}$ do not become significantly greater than unity (in Planckian units) while $a = e^{\omega}$ remains less than unity. Also, if e^{ϕ} (which is effectively the loop expansion parameter) becomes large, we can expect a significant contribution from higher-order terms. Hence on the basis of dimensional analysis alone we expect that the two-loop terms will be at worst roughly equal to the one-loop terms if

$$\dot{\omega}, \quad \dot{\phi}, \quad \dot{\sigma}, \quad e^{\phi}, \quad 1/a < 1 \tag{31}$$

at all times. Note that this is typically a harsher constraint than requiring that the curvature invariants do not exceed the Planck scale.

Having established a heuristic criterion for the importance of higher order terms, we also wish to establish whether bouncing solutions require a special choice of initial conditions. Given λ and δ, a solution to the equations of motion is fully specified by five initial conditions, the sixth being fixed by the constraint. The set of points in "initial conditions space" for which the system evolves towards a particular attractor (that is, a particular asymptotic solution) is known as the "basin of attraction" of the corresponding attractor. The full system of equations is five dimensional, and therefore cannot be easily visualized using a phase space approach. Since the non-singular dynamics depends on the behavior of the modulus field we could set λ to zero (or drop the dilaton entirely) and reduce the system to three dimensions. Previously, however, Cornish and Levin [37] have examined the basins of attraction for two field inflationary models, whose equations of motion are similar to those considered here, by numerically integrating to find the asymptotic behavior for many choices of initial conditions. Using this technique we do not have to restrict ourselves to a reduced system in which the dilaton is trivial, and we can investigate the basin of attraction for solutions where ω and σ have the asymptotic form of equations (20) and (21) and the dilaton tends towards a constant.

In Figure 3 we display the asymptotic form of the solutions derived from a sequence of two dimensional slices through the initial conditions space, where each slice represents a 400×400 grid of $\dot{\omega}_0$ and $\dot{\sigma}_0$ values, for five choices of σ_0 between 0 and 2. Since the equations of motion are unchanged by the transformation $\sigma \to -\sigma$ there is no need to carry out separate integrations for $\sigma_0 < 0$. The integrations were carried out using the Bulirsch-Stoer method [38], with the form of the equations of motion given by equations (14) to (16), while the constraint was used to check the accuracy of the numerical routines.

The intersection of the basins of attraction for a universe that is non-singular as $t \to \pm\infty$ is shown in Figure 3. The intersection of these two attractors defines the region of initial conditions space for which leads to a non-singular, bouncing cosmology. Since this is a substantial volume of the total range of initial conditions, it follows that the universe does not have to be "fine tuned" in order to ensure that it is non-singular. Furthermore, the results obtained here do not depend strongly on ϕ_0 or $\dot{\phi}_0$, or the magnitudes λ and δ.

As well as displaying the choices of initial data corresponding to a bouncing solution, Figure 3 shows the subset of points for which the higher-loop terms in the perturbative expansion are not expected to make a substantial contribution. We see that this in turn represents a large subset of the total basin of attraction, suggesting that the existence of non-singular solutions is not a quirk of the one-loop action. Thus it is clear that the non-singular solutions are in no way special, and that a substantial subset of them are such that the values of a, ϕ, σ and their corresponding velocities are sufficiently small at all times to ensure that the action is unlikely to be dominated by higher-loop corrections.

5 Discussion

In this paper we present explicit non-singular cosmological solutions derived from the one-loop superstring action. These solutions represent a FRW spacetime where the scale factor makes a transition from contraction to expansion, while remaining non-zero at all times. As such, they resemble the type of branch-changing solution that is a prerequisite for the successful implemen-

tation of the pre-big-bang scenario, and demonstrate that although branch-changing does not occur at tree level in the superstring action, non-singular "bounce" solutions do exist at one-loop.

While the non-singular solutions described here occur only when the spatial hypersurfaces have positive curvature and the parameter δ in the one-loop action is negative, our numerical calculations establish that the non-singular solutions do not require a careful choice of initial values for the scale factor, dilaton and modulus fields. Furthermore, while we have not made a detailed analysis of the two-loop terms we expect that their contribution will not dominate the one-loop terms, providing cause for cautious optimism that these non-singular solutions will not vanish when higher loop terms are incorporated into the action.

These solutions allow us to address the initial conditions problem peculiar to the $k = 1$ FRW universe, which typically has a lifetime on the order of the Planck scale. While GUT-scale inflation will solve the other difficulties faced by the standard model of the big bang, inflation can only prevent a closed FRW universe from recollapsing if it commences at the Planck scale. However, we have seen here that for a wide range of initial conditions the equations of motion derived from the one-loop string action permit a closed FRW universe to expand indefinitely. Consequently, this provides a string motivated solution to the version of the curvature problem faced by a $k = 1$ universe without requiring that inflation commences at the Planck scale.

The inflationary epoch cannot be described within the context of the model discussed here, as we have not included matter fields in the action that can drive inflation followed by a graceful exit to a non-inflationary universe with the matter content we observe at low energies. Thus, while our solutions describe a universe which grows arbitrarily large, we know that in practice the assumptions that underpin the action we consider must break down at sufficiently low energy scales. Furthermore, the existence of solutions that describe a closed FRW universe which grows arbitrarily large require that the modulus field is also evolving continuously, which is made possible by the fact the modulus field contains no explicit mass terms at tree level. In a realistic theory the moduli fields must become massive after supersymmetry breaking, which puts a lower limit on the validity of our action even in the

absence of other matter fields and non-perturbative effects. Thus attempts to use the properties of these solutions in a more realistic cosmological model must provide a mechanism that will allow a “graceful exit” from the modulus dominated expansion into an inflationary phase. Furthermore, the late time evolution of the dilaton must be small enough to ensure that the constraints on the time variation of G, the Newtonian gravitational constant, are obeyed. We do not address this problem here, but note that this difficulty afflicts almost all string inspired models, and is not peculiar to the specific example considered here.

Physically, we have seen that the key difference between the one-loop and tree-level actions that permits the existence of non-singular solutions is the coupling between the modulus and the Gauss-Bonnet combination. However, the presence of this term alone cannot guarantee a non-singular universe. Firstly, while the parameter δ can take on both positive and negative values, non-singular solutions only arise when $\delta < 0$. Secondly, even if δ is negative, non-singular evolution is not guaranteed. However, the existence of non-singular solutions to the one-loop action represents a significant improvement on the results found at tree-level and shows that the perturbative limit of string theory can address the singularity problem that is characteristic of the standard big-bang.

Generically, extensions to Einstein gravity consisting of combinations of scalar fields and higher order curvature terms will not lead to a singularity-free theory, but are more likely to make the existing singularity problems worse. Thus it is noteworthy that string theory naturally leads to a low-energy action in which non-singular solutions are possible. More general theories have been considered by Brandenberger and Mukhanov *et al.* [35, 36] who derive a higher order gravitational action for which all homogeneous and isotropic solutions are non-singular. In doing so, they show that this requirement places a strong constraint on the possible form of the action, in the absence of fundamental scalar fields. If fundamental scalar fields were incorporated into their approach, it would be possible to investigate whether models containing second order curvature invariants (such as the Gauss-Bonnet combination) coupled to scalar fields generically permit bouncing solutions and to what extent this is a special property of the one-loop superstring action. Conversely, Rizos and Tamvakis

[39] show that with for a scalar field coupled to the Gauss-Bonnet term with a spatially flat metric, non-singular solutions are possible for a wider range of couplings than the function $\xi(\sigma)$ that is derived from string theory, and it would be of interest to extend their analysis to the $k = 1$ case.

The fact that incorporating string loop contributions into the action leads to a non-singular cosmology is reminiscent of previous attempts to solve the singularity problem by including higher-order terms derived from quantum corrections to the gravitational action [40, 41, 42], which have long been known to remove, or at least soften, the singularity associated with the standard model of the big bang.

Superstring theory currently offers us a tantalizing glimpse of a paradigm that will provide a unified description of fundamental physics which modifies the predictions of general relativity in such a way as to remove the initial singularity associated with the big bang. At present it is impossible to know whether this promise will be fulfilled, but there are strong indications that string theory addresses most, if not all, of the problems of the conventional big bang that can be traced back to the Planck scale.

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References

[1] E. S. Fradkin and A. A. Tseytlin, Phys. Lett. B **158**, 316 (1985).

[2] C. G. Callan, D. Friedan, E. J. Martinec, and M. J. Perry, Nucl. Phys. B **262**, 593 (1985).

[3] C. Lovelace, Nucl. Phys. B **273**, 413 (1986).

[4] D. Bailin and A. Love, Phys. Lett. B **163**, 135 (1985).

[5] K. Maeda, Phys. Rev. D **35**, 471 (1987).

[6] I. Antoniadis, C. Bachas, J. Ellis, and D. V. Nanopoulos, Phys. Lett. B **211**, 393 (1988).

[7] I. Antoniadis, C. Bachas, J. Ellis, and D. V. Nanopoulos, Nucl. Phys. B **328**, 117 (1989).

[8] A. R. Liddle, R. G. Moorhouse, and A. B. Henriques, Nucl. Phys. B **311**, 719 (1989).

[9] M. Mueller, Nucl. Phys. B **337**, 37 (1990).

[10] I. Antoniadis, C. Bachas, J. Ellis, and D. V. Nanopoulos, Phys. Lett. B **257**, 278 (1991).

[11] G. Veneziano, Phys. Lett. B **265**, 287 (1991).

[12] A. A. Tseytlin, Mod. Phys. Lett. A **6**, 1721 (1991).

[13] J. A. Casas, J. García-Bellido, and M. Quirós, Nucl. Phys. B **361**, 713 (1991).

[14] J. García-Bellido and M. Quirós, Nucl. Phys. B **385**, 558 (1992).

[15] A. A. Tseytlin, Class. Quantum Grav. **9**, 979 (1992).

[16] A. A. Tseytlin and C. Vafa, Nucl. Phys. B **372**, 443 (1992).

[17] R. Brustein and G. Veneziano, Phys. Lett. B **329**, 429 (1994).

[18] S. Behrndt and S. Förste, Nucl. Phys. B **430**, 441 (1994).

[19] D. S. Goldwirth and M. J. Perry, Phys. Rev. D **49**, 5019 (1994).

[20] E. J. Copeland, A. Lahiri, and D. Wands, Phys. Rev. D **50**, 4868 (1994).

[21] R. Easther, K. Maeda, and D. Wands, Phys. Rev. D **53**, 4247 (1996).

[22] M. Gasperini and G. Veneziano, Astropart. Phys. **1**, 317 (1993).

[23] M. Gasperini and G. Veneziano, Phys. Rev. D **50**, 2519 (1994).

[24] N. Kaloper, R. Madden, and K. A. Olive, Nucl. Phys. B **452**, 677 (1995).

[25] N. Kaloper, R. Madden, and K. A. Olive, hep-th/9510117 (1995).

[26] M. Gasperini, J. Maharana, and G. Veneziano, hep-th/9602087 (1996).

[27] J. E. Lidsey, gr-qc/9605017 (1996).

[28] I. Antoniadis, J. Rizos, and K. Tamvakis, Nucl. Phys. B **415**, 497 (1994).

[29] E. W. Kolb and M. S. Turner, *The Early Universe*, Vol. 69 of *Frontiers in Physics* (Addison Wesley, Redwood City, California, 1990).

[30] I. Antoniadis, E. Gava, and K. S. Narain, Nucl. Phys. B **383**, 93 (1992).

[31] I. Antoniadis, E. Gava, and K. S. Narain, Phys. Lett. B **283**, 209 (1992).

[32] A. Erdélyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi, *Higher Transcendental Functions* (MacGraw-Hill, New York, 1955), Vol. 3.

[33] L. F. Abbott and M. B. Wise, Nucl. Phys. B **244**, 541 (1984).

[34] G. F. R. Ellis, D. H. Lyth, and M. Mijić, Phys. Lett. B **271**, 52 (1991).

[35] V. F. Mukhanov and R. H. Brandenberger, Phys. Rev. Lett. **68**, 1969 (1992).

[36] R. Brandenberger, V. Mukhanov, and A. Sornborger, Phys. Rev. D **48**, 1629 (1993).

[37] N. J. Cornish and J. J. Levin, Phys. Rev. D **53**, 3022 (1996).

[38] W. H. Press, S. A. Teukolsky, W. T. Vettering, and B. P. Flannery, *Numerical Recipes in FORTRAN*, 2 ed. (Cambridge University Press, Cambridge, 1992).

[39] J. Rizos and K. Tamvakis, Phys. Lett. B **326**, 57 (1994).

[40] H. Nariai, Prog. Theor. Phys. **46**, 433 (1971).

[41] H. Nariai and K. Tomita, Prog. Theor. Phys. **46**, 776 (1971).

[42] V. T. Gurovich, Sov. Phys. JETP **46**, 193 (1977).

Figure Captions

Figure 1: A particular non-singular solution is displayed, where $\delta = -16 \times 3/\pi$ and $\lambda = 1$. The chosen initial values are $\sigma_0 = 0$, $\phi_0 = -3$, $\dot{\phi}_0 = -5 \times 10^{-4}$, $\dot{\sigma}_0 = 0.08$ and $\dot{\omega}_0 = 0.01$, so that the constraint requires $\omega_0 = 3.25114$, and all quantities are expressed in Planckian units. The scale factor passes through a minimum value of approximately 10, and the solution approaches the asymptotic form of equations (24) and (25) as $t \to \pm\infty$.

Figure 2: The value of the scalar, $R_{\mu\nu\kappa\lambda}R^{\mu\nu\kappa\lambda}$ is plotted for the solution depicted in Fig. 1, demonstrating that it remains well below the Planck scale at all times. Note the different scale on the time axis in this plot.

Figure 3: Five different slices through the initial conditions space are displayed, with $\phi_0 = -3$, $\dot{\phi}_0 = -0.005$, $\lambda = 1$ and $\delta = -16 \times 3/\pi$. Note that not all choices of initial conditions correspond to a universe with $k = 1$. The set of initial conditions leading to a non-singular universe constitutes the basin of attraction for the bouncing solutions. The regions inside the solid lines correspond to solutions which do not violate the approximate bounds on the validity of the one-loop approximation, equation (31).

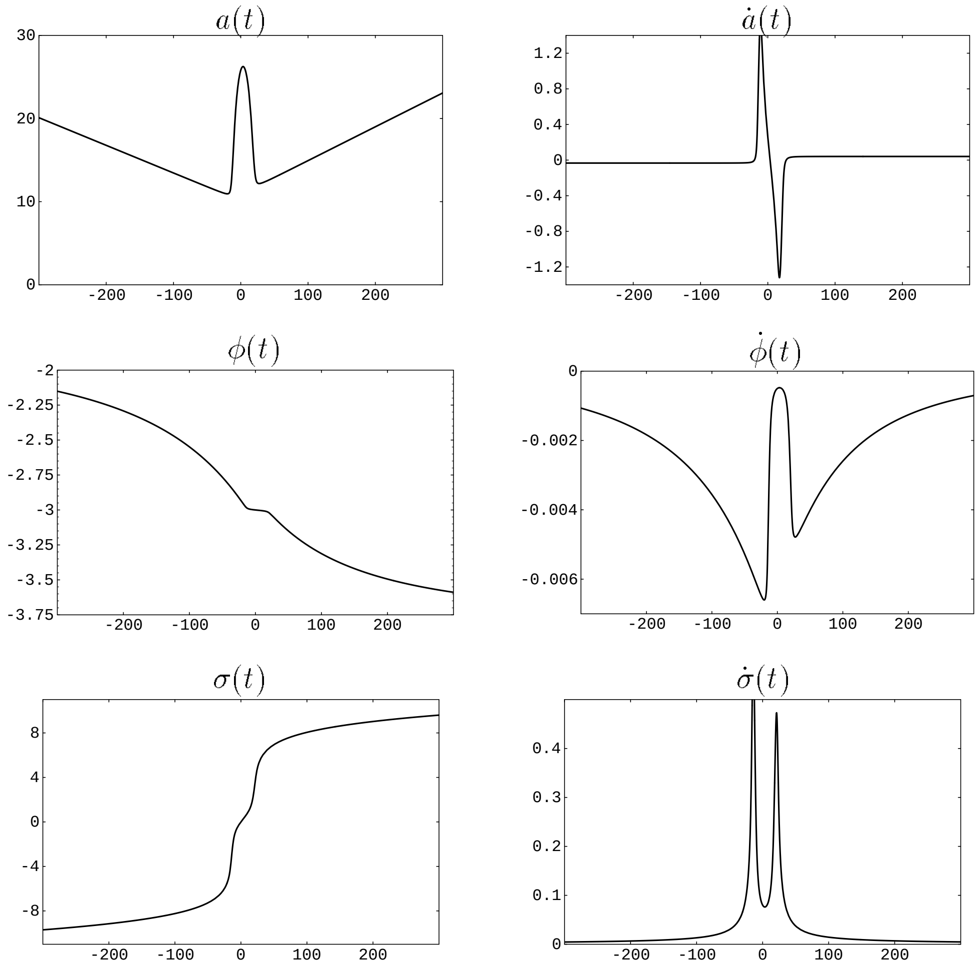

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Figure 1

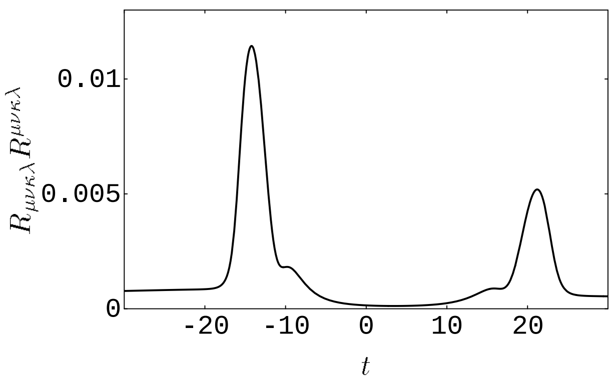

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Figure 2